Experimental Joint Weak Measurement on a Photon Pair as a Probe of Hardy's Paradox

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It has been proposed that the ability to perform joint weak measurements on postselected systems would allow us to study quantum paradoxes. These measurements can investigate the history of those particles that contribute to the paradoxical outcome. Here we experimentally perform weak measurements of joint (i.e., nonlocal) observables. In an implementation of Hardy's paradox, we weakly measure the locations of two photons, the subject of the conflicting statements behind the paradox. Remarkably, the resulting weak probabilities verify all of these statements but, at the same time, resolve the paradox.

Retrodiction is a controversial topic in quantum mechanics [1]. How much is one allowed to say about the history (e.g., particle trajectories) of a postselected ensemble? Historically, this has been deemed a question more suitable for philosophy (e.g., counterfactual logic) than physics; since the early days of quantum mechanics, the standard approach has been to restrict the basis of our physical interpretations to direct experimental observations. On the practical side of the question, postselection has recently grown in importance as a tool in fields such as quantum information: e.g., in linear optics quantum computation [2], where it drives the logic of quantum gates, and in continuous variable systems, for entanglement distillation [3]. Weak measurement is a relatively new experimental technique for tackling just this question. It is of particular interest to carry out weak measurements of multiparticle observables, such as those used in quantum information. Here we present an experiment that uses weak measurement to examine the two-particle retrodiction paradox of Hardy [4,5], confirming the validity of certain retrodictions and identifying the source of the apparent contradiction.

Hardy's paradox is a contradiction between classical reasoning and the outcome of standard measurements on an electron E and positron P in a pair of Mach-Zehnder interferometers (see Fig. 1). Each interferometer is first aligned so that the incoming particle always leaves through the same exit port, termed the "bright" port B (the other is the "dark" port D). The interferometers are then arranged so that one arm (the "inner" arm I) from each interferometer overlaps at Y. It is assumed that if the electron and positron simultaneously enter this arm, they will collide and annihilate with 100% probability. This makes the interferometers "interaction-free measurements" (IFMs) [6]; that is, a click at the dark port indicates the interference was disturbed by an object located in one of the interferometer arms, without the interfering particle itself having traversed that arm. Therefore, in Hardy's paradox a click at the dark port of the electron (positron) interferometer indicates that the positron (electron) was in the inner arm. Consider if one were to detect both particles at the dark

ports. As IFMs, these results would indicate the particles were simultaneously in the inner arms and, therefore, should have annihilated. But this is in contradiction to the fact that they were actually detected at the dark ports. Paradoxically, one does indeed observe simultaneous clicks at the dark ports [7], just as quantum mechanics predicts.

Weak measurements have been performed in classical optical experiments [8], as well as on the polarization of single photons [9]. Weak measurements of joint observables are particularly important, as this class of observables includes nonlocal observables, which can be used to create and identify multiparticle entanglement (e.g., in cluster state computing [10]). Joint observables also include sequential measurements on a single particle, allowing them to characterize time evolution in a system [11]. In this experiment, we demonstrate a new technique that for the first time enables us to perform joint weak measurements.
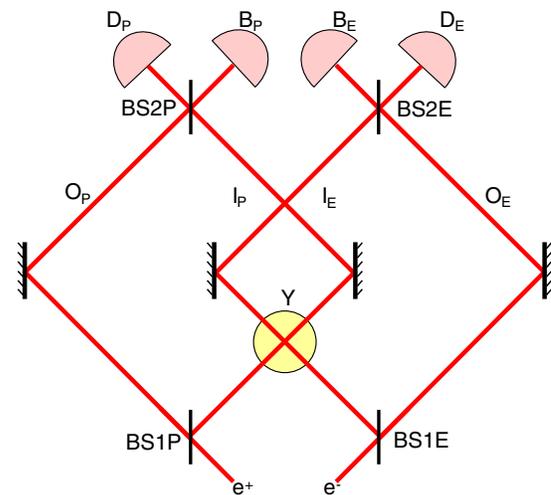

FIG. 1 (color online). Hardy's paradox setup. E and P indicate electron and positron, respectively, which can collide in region Y. BS1P, BS1E, BS2P, and BS2E are 50:50 beam splitters. D and B are the dark and bright ports of the interferometers, respectively. I and O are the inner and outer interferometer arms, respectively.

With this technique, we implement a proposal by
Aharonov *et al.* [5] to weakly measure the simultaneous
location of the two path-entangled photons in Hardy's
paradox [4]. This technique opens up the possibility of
in situ interrogation and characterization of complex multi-
particle quantum systems such as those used in quantum
information.

A standard measurement collapses the measured system,
irreversibly destroying the original quantum state of the
system. Postselected subensembles are particularly diffi-
cult to investigate since measurements on the ensemble
before the postselection will collapse the system and, thus,
alter the action of the postselection itself. Weak measure-
ment was devised by Aharonov, Albert, and Vaidman as a
way of circumventing these problems [12]. It is an exten-
sion of the standard von Neumann measurement model
[13] in which the coupling g between the measured system
and the measurement device is made asymptotically small.
This has the drawback of reducing the amount of informa-
tion that one retrieves in a single measurement. The reward
is that the consequent disturbance of the measured system
is correspondingly small. To extract useful information,
one must repeat the measurement on a large ensemble of
identical quantum systems. The average result is called the
"weak value," denoted $\langle \hat{C} \rangle_W$, where $\hat{C}$ is the measured
operator.

To set up Hardy's paradox, we use two photons instead
of the electron and positron. The experimental setup is
shown in Fig. 2. A diode laser produces a 30 mW
405 nm beam (blue dashed line) which is filtered by a
blue glass filter (BF) and sent through a dichroic mirror
(DM). This beam produces 810 nm collinear photon pairs
(red solid line) in a 4 mm long β-barium borate (BBO)
crystal through the process of type II spontaneous para-
metric down-conversion. These pairs, consisting of a ver-
tical (E) photon and a horizontal (P) photon, take the place
of the electron and positron. The pump passes through a
second DM, to later be retroreflected. The photon pair
passes through a filter (F) to remove any residual pump
light, followed by a 2 mm thick BBO crystal (CC), to
compensate for the birefringent delay in the first crystal.
The photon pair then meets a 50/50 beam splitter
(BS1EP), which acts as the first beam splitter in both the
E and P interferometers, so that each photon can either be
retroreflected and enter the inner arm or be transmitted and
enter the outer arm.

In place of electron-positron annihilation, a quantum
interference effect acts as an absorptive two-photon switch
(Y) [14]. Photons reflected into the inner arm pass back
through the BBO crystal along with the retroreflected
pump beam. The amplitude for the retroreflected pump to
create a pair of photons in the crystal is set to interfere
destructively with the amplitude for a photon pair in the
inner arms. Thus, if both the E and P photons enter their
inner arms, they are removed, whereas if only a single
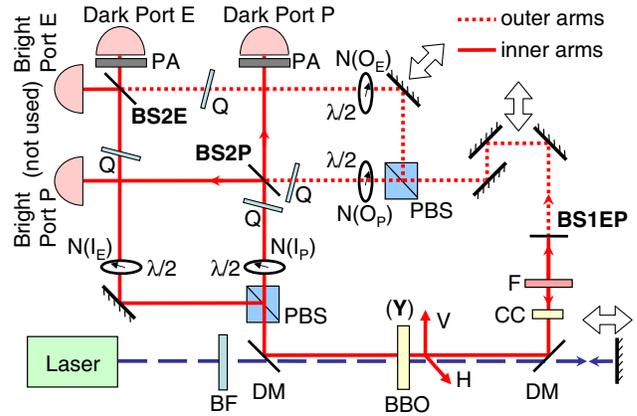photon enters, it passes through the crystal unimpeded. We

FIG. 2 (color online). The experimental setup for the weak
measurement of arm occupations N in Hardy's paradox. An
orthogonally polarized collinear photon pair is produced in the
BBO crystal. Each photon can be either reflected or transmitted
at beam splitter BS1EP (entering the inner or outer arms,
respectively). Pair annihilation in the inner arms is achieved
by an absorptive switching effect at Y. PBSs then separate the
collinear photons into their own interferometers, E and P. Weak
measurements of N are induced with half-wave plates ($\lambda/2$) in
the chosen arms and read out by PAs at the interferometer dark
ports. See the text for more details.

overlap the temporal modes of the two amplitudes by
making the path of the pump from the crystal to the mirror
and back to the crystal equal in length to the analogous path
of the photons reflected by BS1EP. Fine adjustment of this
length sets the phase difference of the amplitudes to be Pi
radians, thereby eliminating $85 \pm 3\%$ of the pairs of pho-
tons in the inner arms.

Photons transmitted at the first beam splitter enter the
outer arms, which contain a variable delay. Next, both the
inner and outer paths encounter polarizing beam splitters
(PBSs) so that the E and P photons are split into their own
spatially separate interferometers. The E interferometer
contains an additional variable delay so that both interfer-
ometers can be adjusted to have the same path-length
difference. The inner (solid line) and outer (dotted line)
paths of the two interferometers are recombined at two
nonpolarizing beam splitters (BS2E and BS2P), taking the
place of the final Mach-Zehnder beam splitters for the
electron and positron. Tilted quartz pieces (Q) before and
after BS2E and P compensate for undesired polarization
phase shifts in them.

Placing a half-wave plate in an arm allows us to measure
whether a photon travelled through this arm. To understand
how this functions, consider a half-wave plate placed in the
E outer arm aligned so as rotate the polarization of a
photon passing through it by 90°. The polarization of the
photon arriving at the E dark port then perfectly indicates if
it was in the E outer arm or not. This a measurement of the
"occupation" $\hat{N}(M_K)$ of the $M = I$ or O (inner or outer)
interferometer arm by photon $K = E$ or P [e.g., $\hat{N}(O_E) =
|O_E\rangle\langle O_E|$]. Unfortunately, this procedure is a standard
projective measurement and, hence, strongly disturbs the

system. In particular, the interference will be destroyed as the two paths are now completely distinguishable, and, thus, the interferometer will not function as an IFM. The strength of the measurement interaction $[\hat{U} = \exp(-ig\hat{N}\hat{\sigma}_y)]$ is parametrized by $g \approx \theta$, the polarization rotation. In this experiment, we reduce this disturbance by rotating the photon's polarization by only 20°, reducing g fourfold, and thereby performing a weak measurement. The trade-off is that it is now impossible to know through which arm a particular detected photon went. Instead, we measure the average polarization rotation at the detector over many trials to find what fraction of photons passed through that particular arm. If no rotation is observed, then the classical inference would be that the photon was never in the arm with the wave plate. Conversely, if we measure an average rotation of 20°, one might infer that every photon passed through the wave plate. Quantum mechanically, this rotation constitutes a weak measurement of the occupation $\hat{N}$ of a particular interferometer arm.

The crux of the paradox is that the detected photons cannot have simultaneously been in the inner arms. To test this we require a weak measurement of the *joint* occupation of two arms. It was previously thought that a physical interaction between the particles was necessary to make weak measurements of joint observables (e.g., the electrostatic interaction of ions, as in Ref. [15]). In Ref. [16], it was shown theoretically that one needs only to perform single-particle weak measurements on each particle. The joint weak values then appear in polarization correlations between the two particles as follows:

$$\langle \hat{N}(M_K)\rangle_W = g^{-1}\text{Re}\langle\hat{\sigma}_{zK}^-\rangle, \qquad (1)$$

$$\langle \hat{N}(M_P)\hat{N}(M_E)\rangle_W = g^{-2}\text{Re}\langle\hat{\sigma}_{zP}^-\hat{\sigma}_{zE}^-\rangle, \qquad (2)$$

where $\hat{\sigma}_{zK}^- = (\hat{\sigma}_{xK} - i\hat{\sigma}_{yK})$ is the z basis lowering operator for the polarization of photon $K = E$ or P. We weakly measure all four combinations of $\hat{N}(M_E)\hat{N}(M_P)$ by placing half-wave plates ($\lambda/2$) in all four arms just before the final beam splitters. We measure the occupation of a particular pair of arms by rotating only those two wave plates. After the final beam splitters, we measure average polarization rotations as well as the correlations specified in Eq. (2) with polarization analyzers (PAs) consisting of a quarter-wave plate and polarizer followed by a single-photon detector (Perkin Elmer SPCM-AQR). Once the Pauli operators are substituted in Eq. (2) and the real part is found, four Pauli operators remain in the final expectation value.

For each of these Pauli operators, the analyzer must be set to two positions [e.g., for $\hat{\sigma}_x$, 45° and −45° (↗ and ↘) and, for $\hat{\sigma}_y$, right-hand circular and left-hand circular (↻ and ↺)]. Thus, each joint weak value requires eight measurements of coincidence rates at the two dark ports:

$$\text{Re}\langle\hat{\sigma}_{zP}^-\hat{\sigma}_{zE}^-\rangle = \frac{R_{↗↗} + R_{↘↘} - R_{↘↗} - R_{↗↘}}{R_{↗↗} + R_{↘↘} + R_{↘↗} + R_{↗↘}}$$
$$- \frac{R_{↻↻} + R_{↺↺} - R_{↺↻} - R_{↻↺}}{R_{↻↻} + R_{↺↺} + R_{↺↻} + R_{↻↺}}, \qquad (3)$$

where R_{sq} is the coincidence rate when the $P(E)$ analyzer is set to $s(q)$. Single weak values for the occupation of interferometer $P(E)$ are found from these rates by summing over analyzer settings for photon $E(P)$.

Table I gives the full set of measured coincidence rates needed to determine the weak values discussed above. The first two columns indicate which arm occupations are being weakly measured. The next eight columns give the coincidence rate per 420 s for the various polarization analyzer settings. Errors in the rates are assumed to be Poissonian, i.e., $\Delta R = \sqrt{R}$. The last two columns give $g_{E(P)}$ (with error ± 0.003), which was independently measured for each arm to account for polarization-dependent losses, and $g^2 = g_E \cdot g_P$.

In Table II, we present the resulting weak values for the various arm occupations. The bottom cells and rightmost cells give the weak value for the occupation of a single arm, and the inner cells give the joint occupation of a pair of arms. Error bars are derived from uncertainties in g and an assumption of Poissonian statistics in the rates.

Examining the table reveals that the single-particle weak measurements are consistent with the clicks at each dark port; as the IFM results imply, the weakly measured occupations of each of the inner arms are close to 1 and those of each of the outer arms are close to zero. The weak measurements indicate that, at least when considered individually, the photons were in the inner arms. However, if we instead examine the joint occupation of the two inner arms, it appears that the two photons are simultaneously present only roughly one-quarter of the time. This demonstrates that, as we expect, the particles are not in the inner arms together.

So far, we seem to have confirmed both of the premises of Hardy's paradox: to wit, that when D_P and D_E fire, $N(I_P)$ and $N(I_E)$ are close to 1 (since the IFMs indicate the presence of the particles in Y)—but that $N(I_P \& I_E)$ is close to zero (since, when *both* particles are in Y, they annihilate

TABLE I. The measured coincidence rates needed to determine the weak values.

E	P	$R_{↗↗}$	$R_{↗↘}$	$R_{↘↘}$	$R_{↘↗}$	$R_{↻↻}$	$R_{↺↺}$	$R_{↺↻}$	$R_{↻↺}$	g_E	g_P
O	O	556	834	583	730	750	543	666	571	0.674	0.541
I	I	2261	772	115	746	1030	762	913	729	0.635	0.570
I	O	1152	1079	351	179	484	655	452	654	0.635	0.541
O	I	1051	260	329	769	715	609	388	825	0.674	0.570

TABLE II. The weak values for the arm occupations in Hardy's paradox.

	$N(I_E)$	$N(O_E)$	
$N(I_P)$	0.245 ± 0.068 [0]	0.641 ± 0.083 [1]	$\mathbf{0.926 \pm 0.015}$ **[1]**
$N(O_P)$	0.719 ± 0.074 [1]	-0.759 ± 0.083 [-1]	$\mathbf{-0.078 \pm 0.02}$ **[0]**
	$\mathbf{0.924 \pm 0.024}$ **[1]**	$\mathbf{0.087 \pm 0.023}$ **[0]**	

and should not be detected). This is odd because in classical logic, $N(I_P \& I_E)$ must be $\geq N(I_P) + N(I_E) - 1$; this inequality is violated by our results. Although $N(I_P)$ is 93% and $N(I_E)$ is 92%, the data in Table II suggest that when E is in the inner path, P is not, and vice versa, hence the large values for $N(I_P \& O_E) = 64\%$ and $N(O_P \& I_E) = 72\%$. The fact that the sum of these two seemingly disjoint joint-occupation probabilities exceeds 1 is the contradiction with classical logic. In the context of weak measurements, the resolution of this problem lies in the fact that weak valued probabilities are not required to be positive definite [5], and so a negative occupation $N(O_P \& O_E) = -76\%$ is possible, preserving the probability sum rules. In an ideal implementation of Hardy's paradox, the joint probabilities are strictly 0 for both particles to be in inner arms, -1 for both to be in the outer, and 1 for either to be in the inner while the other is in the outer arm. These are indicated in brackets in Table II, for comparison with our experimental data. Discrepancies are because of the imperfect switch efficiency ($85 \pm 3\%$) and IFM probabilities ($95 \pm 3\%$ for the P IFM and $94 \pm 4\%$ for E).

What is the meaning of the negative joint occupation? Recall that the joint values are extracted by studying the polarization rotation of both photons in coincidence. Consider a situation in which both photons always simultaneously passed through two particular arms. When a polarization rotator is placed in each of these arms, it would tend to cause their polarizations to rotate in a correlated fashion; when P was found to have 45° polarization, E would also be more likely to be found at 45° than $-45°$. Experimentally, we find the reverse—when P is found to have 45° polarization, E is preferentially found at $-45°$ (and vice versa), as though it had rotated in the direction opposite to the one induced by the physical wave plate. As in all weak-measurement experiments, a negative weak value implies that the shift of a physical "pointer" (in this case, photon polarization) has the opposite sign from the one expected from the measurement interaction itself.

In summary, Hardy's paradox is a set of conflicting classical logic statements about the location of the particles in each of two Mach-Zehnder interferometers. It is impossible to simultaneously verify these statements with standard measurements since testing one statement disturbs the system and consequently nullifies the other statements. We attempt to minimize this disturbance by reducing the strength of the interaction used to perform the measurement. The results of these weak measurements indicate that all of the logical statements are correct and also provide a

self-consistent, if strange, resolution to the paradox. Since they do not disturb subsequent postselection of the systems under study, weak measurements are ideal for the interrogation and characterization of postselected multiparticle states such as cluster states and processes such as linear optics quantum computation. This experiment demonstrates a new technique that, for the first time, allows for the weak measurement of general multiparticle observables in these systems.

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